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On integrals and invariants for inviscid, irrotational flow under gravity

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Simplified proofs are given of the eight basic conservation laws for two-dimensional irrotational flow of a perfect fluid. The corresponding relations for rotational flow are also discussed.

1. Introduction

In a recent paper Benjamin & Olver (1983) have enumerated a number of conserved properties associated with the equations for a perfect fluid in irrotational motion under the action of gravity and surface tension. Their derivation is given in a rather general form, and is based in turn on some results of Olver (1980) for the possible sets of integral invariants of evolution equations under the transformations of a Lie group. In particular, when the motion is two-dimensional and surface tension is absent, Benjamin & Olver showed the existence of just eight conserved 'densities', each with a corresponding flux. Some of these, for example the densities of mass, momentum and energy, are well known; others, such as the angular momentum, and another similar-looking quantity, are less familiar.

Because of the generality of Benjamin & Olver's analysis and the length of their derivations, the present author has found it useful to deduce the particular results just mentioned in a more simple and direct way. The method employed is indeed referred to passingly in §6 of their paper. However, it may be useful to give these alternative proofs explicitly and from first principles, so that the great gain in clarity and simplicity becomes evident.

This is the aim of the following note. For the deeper question of *completeness* of the integral properties, the reader will, of course, have to study the arguments of Benjamin & Olver (1983), which paper in turn refers to the more rigorous work of Olver.

Some generalizations of the results to rotational motion are given in §6.

2. Definitions

Let x and y be rectangular coordinates in the plane of motion, with y vertically upwards, and let ϕ denote the velocity potential. If C denotes any simple closed contour bounding a domain D of the fluid, then using Green's identity

$$\iint_D (P_y - Q_x)\,\mathrm{d}x\,\mathrm{d}y = \int_C (P\,\mathrm{d}x + Q\,\mathrm{d}y) \tag{2.1}$$

(where suffixes denote partial differentiation) we may define the following eight integral quantities:

$$M = \iint_D \mathrm{d}x\,\mathrm{d}y = \int_C y\,\mathrm{d}x, \tag{2.2}$$

$$\begin{cases} M\bar{x} = \iint_D x\,\mathrm{d}x\,\mathrm{d}y = \int_C xy\,\mathrm{d}x, & (2.3) \\ M\bar{y} = \iint_D y\,\mathrm{d}x\,\mathrm{d}y = \int_C \tfrac{1}{2}y^2\,\mathrm{d}x, & (2.4) \end{cases}$$

$$\begin{cases} I = \iint_D \phi_x\,\mathrm{d}x\,\mathrm{d}y = \int_C (-\phi)\,\mathrm{d}y, & (2.5) \\ J = \iint_D \phi_y\,\mathrm{d}x\,\mathrm{d}y = \int_C \phi\,\mathrm{d}x, & (2.6) \end{cases}$$

$$\begin{cases} A = \iint_D [(x\phi)_y - (y\phi)_x]\,\mathrm{d}x\,\mathrm{d}y = \int_C \phi(x\,\mathrm{d}x + y\,\mathrm{d}y), & (2.7) \\ B = \iint_D [(x\phi)_x + (y\phi)_y]\,\mathrm{d}x\,\mathrm{d}y = \int_C \phi(y\,\mathrm{d}x - x\,\mathrm{d}y), & (2.8) \end{cases}$$

$$T = \iint_D \tfrac{1}{2}(\phi_x^2 + \phi_y^2)\,\mathrm{d}x\,\mathrm{d}y = \int_C \phi(\phi_y\,\mathrm{d}x - \phi_x\,\mathrm{d}y). \tag{2.9}$$

In deriving (2.9) we use in addition the fact that $\nabla^2\phi = 0$, by continuity. If we define also

$$E = T + V, \quad \text{where} \quad V = Mg\bar{y}, \tag{2.10}$$

then the eight quantities (2.2)–(2.8) and (2.10) correspond to the eight quantities I_3, I_5, I_6, I_1, I_4, I_8, I_7 and I_2 in Benjamin & Olver (1983, §6)†. Thus M is familiar as the total mass (the fluid density being taken as unity); $\bar{x}$ and $\bar{y}$ are the coordinates of the centre of mass; I and J are the two components of the momentum; A is the angular momentum; B a relatively unfamiliar quantity, analogous to A; and E is the total energy, being the sum of the kinetic energy T and the potential energy V.

3. Theorems

Let p denote the pressure as given by Bernoulli's equation

$$p + gy + \tfrac{1}{2}(\phi_x^2 + \phi_y^2) + \phi_t = 0, \tag{3.1}$$

an arbitrary function of the time having been absorbed into ϕ_t. Then the theorems to be proved are as follows. If the contour C moves with the fluid, then

$$\frac{\mathrm{d}M}{\mathrm{d}t} = 0, \tag{3.2}$$

$$\begin{cases} M\dfrac{\mathrm{d}\bar{x}}{\mathrm{d}t} = I, & (3.3) \\ M\dfrac{\mathrm{d}\bar{y}}{\mathrm{d}t} = J, & (3.4) \end{cases}$$

$$\begin{cases} \dfrac{\mathrm{d}I}{\mathrm{d}t} = \int_C p\,\mathrm{d}y, & (3.5) \\ \dfrac{\mathrm{d}J}{\mathrm{d}t} = -\int_C p\,\mathrm{d}x - Mg, & (3.6) \end{cases}$$

† Except that these authors write I^n for I_n.

$$\left\{\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t} &= -\int_C p(x\,\mathrm{d}x + y\,\mathrm{d}y) - Mg\bar{x}, && (3.7)\\ \frac{\mathrm{d}B}{\mathrm{d}t} &= -\int_C p(y\,\mathrm{d}x - x\,\mathrm{d}y) + 4T - 3V, && (3.8)\end{aligned}\right.$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_C p(\phi_y\,\mathrm{d}x - \phi_x\,\mathrm{d}y). \quad (3.9)$$

4. Proofs

Note that since C moves with the fluid, which is incompressible, then in general

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_D R\,\mathrm{d}x\,\mathrm{d}y = \iint_D \frac{\mathrm{D}R}{\mathrm{D}t}\mathrm{d}x\,\mathrm{d}y, \quad (4.1)$$

where $\mathrm{D}/\mathrm{D}t$ denotes differentiation following the motion. To prove (3.2)–(3.4) we have

$$\frac{\mathrm{D}}{\mathrm{D}t}(1) = 0, \quad (4.2)$$

$$\frac{\mathrm{D}x}{\mathrm{D}t} = \phi_x, \quad (4.3)$$

$$\frac{\mathrm{D}y}{\mathrm{D}t} = \phi_y. \quad (4.4)$$

To prove (3.5) and (3.6) note that from (3.1)

$$\frac{\mathrm{D}\phi_x}{\mathrm{D}t} = -p_x, \quad (4.5)$$

$$\frac{\mathrm{D}\phi_y}{\mathrm{D}t} = -p_y - g. \quad (4.6)$$

To prove (3.7) note that

$$\frac{\mathrm{D}\phi}{\mathrm{D}t} = \phi_t + (\phi_x^2 + \phi_y^2) = -(p+gy) + \tfrac{1}{2}(\phi_x^2 + \phi_y^2)$$

by (3.1). Hence using (4.3)–(4.6) we have

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{D}t}[(x\phi)_y - (y\phi)_x] &= \frac{\mathrm{D}}{\mathrm{D}t}(x\phi_y - y\phi_x)\\ &= \left(\frac{\mathrm{D}x}{\mathrm{D}t}\phi_y - \frac{\mathrm{D}y}{\mathrm{D}t}\phi_x\right) + \left(x\frac{\mathrm{D}\phi_y}{\mathrm{D}t} - y\frac{\mathrm{D}\phi_x}{\mathrm{D}t}\right)\\ &= (\phi_x\phi_y - \phi_y\phi_x) - x(p_y + g) + yp_x\\ &= -(xp)_y + (yp)_x - gx, \end{aligned} \quad (4.7)$$

from which (3.7) follows. Similarly

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{D}t}[(x\phi)_x + (y\phi)_y] &= \frac{\mathrm{D}}{\mathrm{D}t}[x\phi_x + y\phi_y + 2\phi]\\ &= (\phi_x^2 + \phi_y^2) - xp_x - y(p_y + g) - 2[(p+gy) - \tfrac{1}{2}(\phi_x^2 + \phi_y^2)]\\ &= -(xp)_x - (yp)_y + 2(\phi_x^2 + \phi_y^2) - 3gy, \end{aligned} \quad (4.8)$$

from which (3.8) follows. Lastly

$$\frac{D}{Dt}[\tfrac{1}{2}(\phi_x^2+\phi_y^2)+gy] = \left(\phi_x\frac{D\phi_x}{Dt}+\phi_y\frac{D\phi_y}{Dt}\right)+g\frac{Dy}{Dt}$$

$$= -\phi_x p_x - \phi_y(p_y+g)+g\phi_y$$

$$= -(p\phi_x)_x-(p\phi_y)_y, \tag{4.9}$$

since $\nabla^2\phi = 0$. Then (3.9) follows.

5. Conserved quantities

From (3.2)–(3.9) it follows immediately that if p vanishes everywhere on the boundary, then the following eight quantities are conserved:

$$M = c_1, \tag{5.1}$$

$$I = c_2, \tag{5.2}$$

$$J+c_1gt = c_3, \tag{5.3}$$

$$M\bar{x}-c_2t = c_4, \tag{5.4}$$

$$M\bar{y}-c_3t+\tfrac{1}{2}gt^2 = c_5, \tag{5.5}$$

$$E = c_6, \tag{5.6}$$

$$A+g(c_4t+\tfrac{1}{2}c_2t^2) = c_7, \tag{5.7}$$

$$B-4c_6t+7g(c_5t+\tfrac{1}{2}c_3t^2-\tfrac{1}{6}c_1gt^3) = c_8. \tag{5.8}$$

In the special case $g = 0$, these quantities reduce to M, I, J, $(M\bar{x}-It)$, $(M\bar{y}-Jt)$, E, A and $(B-4Et)$ respectively.

6. Motion with vorticity

When the motion is not irrotational, there no longer exists a velocity potential ϕ. Nevertheless we may define M, $\bar{x}$ and $\bar{y}$ as in (2.3)–(2.5) and also

$$I = \iint_D u\,dx\,dy, \tag{6.1}$$

$$J = \iint_D v\,dx\,dy, \tag{6.2}$$

$$A = \iint_D (xv-yu)\,dx\,dy, \tag{6.3}$$

$$E = \iint_D [\tfrac{1}{2}(u^2+v^2)+gy]\,dx\,dy. \tag{6.4}$$

There appears to be no simple analogue to B (equation (2.8)). We may, however, define the 'circulation'

$$C = \iint_D (v_y-u_x)\,dx\,dy = \int_C (v\,dx+u\,dy). \tag{6.5}$$

Then from the relations

$$\frac{Dx}{Dt} = u, \quad \frac{Dy}{Dt} = v, \tag{6.6}$$

$$\frac{\mathrm{D}u}{\mathrm{D}t} = -p_x, \quad \frac{\mathrm{D}v}{\mathrm{D}t} = -p_y - g \tag{6.7}$$

together with (4.2), the six equations (3.2)–(3.7) follow as before, while (3.9) is replaced by

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_C p(v\,\mathrm{d}x - u\,\mathrm{d}y). \tag{6.8}$$

There appears no analogue to (3.8). But from the vorticity equation

$$\frac{\mathrm{D}}{\mathrm{D}t}(v_y - u_x) = 0 \tag{6.9}$$

for two-dimensional flow we immediately derive from (6.5) the circulation theorem

$$\frac{\mathrm{d}C}{\mathrm{d}t} = 0. \tag{6.10}$$

We note that the angular-momentum equation (3.7), which we have shown is valid also for rotational flow, can be written in the alternative form

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\int_C (p+gy)(x\,\mathrm{d}x + y\,\mathrm{d}y) \tag{6.11}$$

since the integral of gy^2 round a closed curve vanishes. We have then an alternative derivation of the theorem given by Longuet-Higgins (1980, equation (4.10)).

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